

Remarkable Degenerate Quantum Stabilizer Codes Derived from Duadic Codes

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Abstract—Good quantum codes, such as quantum MDS codes, are typically nondegenerate, meaning that errors of small weight require active error-correction, which is—paradoxically—itself prone to errors. Decoherence free subspaces, on the other hand, do not require active error correction, but perform poorly in terms of minimum distance. In this paper, examples of degenerate quantum codes are constructed that have better minimum distance than decoherence free subspaces and allow some errors of small weight that do not require active error correction. In particular, two new families of $[[n, 1, \geq \sqrt{n}]]_q$ degenerate quantum codes are derived from classical duadic codes.

I. INTRODUCTION

Suppose that q is a power of a prime p . Recall that an $[[n, k, d]]_q$ quantum stabilizer code Q is a q^k -dimensional subspace of \mathbf{C}^{q^n} such that $\langle u|E|u\rangle = \langle v|E|v\rangle$ holds for any error operator E of weight $\text{wt}(E) < d$ and all $|u\rangle, |v\rangle \in Q$, see [1], [7] for details. The stabilizer code Q is called nondegenerate (or pure) if and only if $\langle v|E|v\rangle = q^{-n} \text{tr } E$ holds for all errors E of weight $\text{wt}(E) < d$; otherwise, Q is called degenerate. Recall that purity and nondegeneracy are equivalent notions in the case of stabilizer codes, see [3], [5].

In spite of the negative connotations of the term “degenerate”, we will argue that degeneracy is an interesting and in some sense useful quality of a quantum code. Let us call an error nice if and only if it acts by scalar multiplication on the stabilizer code. Nice errors do not require any correction, which is a nice feature considering the fact that operational imprecisions of a quantum computer can introduce errors in a correction step (which is the main reason why elaborate fault-tolerant implementations are needed).

If we assume a depolarizing channel, then errors of small weight are more likely to occur than errors of large weight. If the stabilizer code Q is nondegenerate, then all nice errors have weight d or larger, so the most probable errors *all* require (potentially hazardous) active error correction. On the other hand, if the stabilizer code is degenerate, then there exist nice errors of weight less than the minimum distance. Given these observations, it would be particularly interesting to find degenerate stabilizer codes with many nice errors of small weight.

Although the first quantum error-correcting code by Shor was a degenerate $[[9, 1, 3]]_2$ stabilizer code, it turns out that most known quantum stabilizer code families provide pure codes. If one insists on a large minimum distance, then nondegeneracy seems more or less unavoidable (for example, quantum MDS codes are necessarily nondegenerate, see [11]).

However, the fact that most known stabilizer codes do not have nice errors of small weight is the result of more pragmatic considerations.

Let us illustrate this last remark with the CSS construction; similar points can be made for other stabilizer code constructions. Suppose we start with a classical self-orthogonal $[[n, k, d]]_q$ code C , then one can obtain with the CSS construction an $[[n, n-2k, \delta]]_q$ stabilizer code, where $\delta = \text{wt}(C^\perp \setminus C)$. Since we often do not know the weight distribution of the code C , the easiest way to obtain a stabilizer code with minimum distance at least δ_0 is to choose C such that its dual distance $d^\perp \geq \delta_0$, as this ensures $\delta \geq d^\perp \geq \delta_0$. However, since $C \subseteq C^\perp$, the side effect is that all nonscalar nice errors have a weight of at least $d \geq d^\perp \geq \delta_0$.

Our considerations above suggest a different approach. Since we would like to have nice errors of small weight, we start with a classical self-orthogonal code C that has a small minimum distance, but is chosen such that the vector of smallest Hamming weight in the difference set $C^\perp \setminus C$ is large. In general, it is of course difficult to find a good lower bound for the weights in this difference set.

We illustrate this approach for degenerate quantum stabilizer codes that are derived from classical duadic codes. Recall that the duadic codes generalize the quadratic residue codes, see [9], [14], [15]. We show that one can still obtain a surprisingly large minimum distance, considering the fact we start with classical codes that are really bad.

In Section II, we recall basic properties of duadic codes. In Section III, we construct degenerate quantum stabilizer codes using the CSS construction. Finally, in Section IV, we obtain further quantum stabilizer codes using the Hermitian code construction.

Notation: Throughout this paper, n denotes a positive odd integer. If a is an integer coprime to n , then we denote by $\text{ord}_n(a)$ the multiplicative order of a modulo n . We briefly write $q \equiv \square \pmod n$ to express the fact that q is a quadratic residue modulo n . We write $p^\alpha \parallel n$ if and only if the integer n is divisible by p^α but not by $p^{\alpha+1}$. If $\text{gcd}(a, n) = 1$, then the map $\mu_a : i \mapsto ai \pmod n$ denotes a permutation on the set $\{0, 1, \dots, n-1\}$. An element $c = (c_1, \dots, c_n) \in \mathbf{F}_q^n$ is said to be even-like if $\sum_i c_i = 0$, and odd-like otherwise. A code $C \subseteq \mathbf{F}_q^n$ is said to be even-like if every codeword in C is even-like, and odd-like otherwise.

II. CLASSICAL DUADIC CODES

In this section, we recall the definition and basic properties of duadic codes of length n over a finite field \mathbf{F}_q such that $\gcd(n, q) = 1$. For each choice, we will obtain a quartet of codes: two even-like cyclic codes and two odd-like cyclic codes.

Let S_0, S_1 be the defining sets of two cyclic codes of length n over \mathbf{F}_q such that

- 1) $S_0 \cap S_1 = \emptyset$,
- 2) $S_0 \cup S_1 = S = \{1, 2, \dots, n-1\}$, and
- 3) $aS_i \bmod n = S_{(i+1 \bmod 2)}$ for some a coprime to n .

In particular, each S_i is a union of q -ary cyclotomic cosets modulo n . Since condition 3) implies $|S_0| = |S_1|$, we have $|S_i| = (n-1)/2$, whence n must be odd. The tuple $\{S_0, S_1, a\}$ is called a *splitting* of n given by the permutation μ_a .

Let α be a primitive n -th root of unity over \mathbf{F}_q . For $i \in \{0, 1\}$, the odd-like duadic code D_i is a cyclic code of length n over \mathbf{F}_q with defining set S_i and generator polynomial

$$g_i(x) = \prod_{j \in S_i} (x - \alpha^j).$$

The even-like duadic code C_i is defined as the even-like subcode of D_i ; thus, it is a cyclic code with defining set $S_i \cup \{0\}$ and generator polynomial $(x-1)g_i(x)$. We have $\dim D_i = (n+1)/2$ and $\dim C_i = (n-1)/2$.

Theorem 1: Duadic codes of length n over \mathbf{F}_q exist if and only if $q \equiv \square \bmod n$.

Proof: This is well-known, see, for example, [15, Theorem 1] or [6, Theorem 6.3.2, pages 220-221]. ■

Although the weight distribution of a duadic code is not known in general, the following well-known fact gives partial information about the weights of odd-like codewords.

Lemma 2 (Square Root Bound): Let D_0 and D_1 be a pair of odd-like duadic codes of length n over \mathbf{F}_q . Then their minimum odd-like weights in both codes are same, say d_o . We have

- 1) $d_o^2 \geq n$,
- 2) $d_o^2 - d_o + 1 \geq n$ if the splitting is given by μ_{-1} .

Proof: See [6, Theorem 6.5.2]. ■

III. QUANTUM DUADIC CODES – EUCLIDEAN CASE

In this section, we derive quantum stabilizer codes from classical duadic code using the well-known CSS construction. Recall that in the CSS construction, the existence of an $[n, k_1]_q$ code C and an $[n, k_2]_q$ code D such that $C \subset D$ guarantees the existence of an $[[n, k_2 - k_1, d]]_q$ quantum stabilizer code with minimum distance $d = \min \text{wt}\{(D \setminus C) \cup (C^\perp \setminus D^\perp)\}$.

A. Basic Code Constructions

Recall that two \mathbf{F}_q -linear codes C_1 and C_2 are said to be equivalent if and only if there exists a monomial matrix M and automorphism γ of \mathbf{F}_q such that $C_2 = C_1 M \gamma$, see [6, page 25]. We denote equivalence of codes by $C_1 \sim C_2$. For us it is relevant that equivalent codes have the same weight distribution, see [6, page 25].

The permutation map $\mu_a : i \mapsto ai \bmod n$ also defines an action on polynomials in $\mathbf{F}_q[x]$ by $f(x)\mu_a = f(x^a)$. This induces an action on a cyclic code C over \mathbf{F}_q by

$$C\mu_a = \{c(x)\mu_a \mid c(x) \in C\} = \{c(x^a) \mid c(x) \in C\}.$$

Lemma 3: Let C be a cyclic code of length n over \mathbf{F}_q with defining set T . If $\gcd(a, n) = 1$, then the cyclic code $C\mu_a$ has the defining set $a^{-1}T$. Furthermore, we have $C\mu_a \sim C$.

Proof: This follows from the definitions, see also [6, Corollary 4.4.5] and [6, page 141]. ■

Theorem 4: Let n be a positive odd integer, and let $q \equiv \square \bmod n$. There exist quantum duadic codes with the parameters $[[n, 1, d]]_q$, where $d^2 \geq n$. If $\text{ord}_n(q)$ is odd, then there also exist quantum duadic codes with minimum distance $d^2 - d + 1 \geq n$.

Proof: Let $N = \{0, 1, \dots, n-1\}$. If $q \equiv \square \bmod n$, then there exist duadic codes $C_i \subset D_i$, for $i \in \{0, 1\}$. Suppose that the defining set of D_i is given by S_i ; thus, the defining set of the even-like subcode C_i is given by $S_i \cup \{0\}$. It follows that C_i^\perp has defining set $-(N \setminus (\{0\} \cup S_i)) = -S_{(i+1 \bmod 2)}$. Using Lemma 3, we obtain $C_i^\perp = D_{(i+1 \bmod 2)}\mu_{-1} \sim D_{(i+1 \bmod 2)}$ and $D_i^\perp = C_{(i+1 \bmod 2)}\mu_{-1} \sim C_{(i+1 \bmod 2)}$. By the CSS construction, there exists an $[[n, (n+1)/2 - (n-1)/2, d]]_q$ quantum stabilizer code with minimum distance $d = \min\{\text{wt}((D_i \setminus C_i) \cup (C_i^\perp \setminus D_i^\perp))\}$. Since $C_i^\perp \sim D_{(i+1 \bmod 2)}$ and $D_i^\perp \sim C_{(i+1 \bmod 2)}$, the minimum distance $d = \min\{\text{wt}((D_i \setminus C_i) \cup (D_{(i+1 \bmod 2)} \setminus C_{(i+1 \bmod 2)}))\}$, which is nothing but the minimum odd-like weight of the duadic codes; hence $d^2 \geq n$. If $\text{ord}_n(q)$ is odd, then μ_{-1} gives a splitting of n [12, Lemma 5]. In this case, Lemma 2 implies that the odd-like weight d satisfies $d^2 - d + 1 \geq n$. ■

In the binary case, it is possible to derive degenerate codes with similar parameters using topological constructions [2], [4], [8], but the codes do not appear to be equivalent to the construction given here.

B. Degenerate Codes

The next result proves the existence of degenerate duadic quantum stabilizer codes. This result shows that the classical duadic codes, such as $C_i \subseteq D_i$, contain codewords of very small weight but their set difference $D_i \setminus C_i$ (and $C_i^\perp \setminus D_i^\perp$) does not. First we need the following lemma, which shows the existence of duadic codes of low distance.

Lemma 5: Let $n = \prod p_i^{m_i}$ be an odd integer and $q \equiv \square \bmod p_i$. If $t_i = \text{ord}_{p_i}(q)$ and $p_i^{z_i} \parallel q^{t_i} - 1$, and $m_i > 2z_i$, then there exists a duadic code of length n and (even-like) minimum distance $\leq \min\{p_i^{z_i}\} < \sqrt{n}$.

Proof: By Theorem 1 there exist duadic codes of lengths $p_i^{m_i}$ and by [15, Theorem 6] their minimum distance, d'_i is less than $p_i^{z_i}$. Since we know that the odd-like distance is $\geq p_i^{m_i/2} > p_i^{z_i}$, the minimum distance must be even-like. By [15, Theorem 4], there exists duadic codes of length $n = \prod p_i^{m_i}$ whose minimum distance $d' \leq \min\{d'_i\} \leq \min\{p_i^{z_i}\} < \prod p_i^{m_i/2} = \sqrt{n}$. Since this is less than the

minimum odd-like distance, the minimum distance is even-like. ■

Theorem 6: Let $n = \prod p_i^{m_i}$ be an odd integer and $q \equiv \square \pmod{p_i}$. Let $t_i = \text{ord}_{p_i}(q)$, and let z_i be such that $p_i^{z_i} \parallel q^{t_i} - 1$. Then for $m_i > 2z_i$, there exists a degenerate $[[n, 1, d]]_q$ quantum code pure to $d' \leq \min\{p_i^{z_i}\} < d$ with $d^2 \geq n$. If $p_i \equiv -1 \pmod{4}$, then $d^2 - d + 1 \geq n$.

Proof: From Lemma 5, we know that there exist duadic codes of length n and minimum (even-like) distance $d' \leq \min\{p_i^{z_i}\} < \sqrt{n}$. From Theorem 4, we know there exists a quantum duadic code with parameters $[[n, 1, d]]_q$, where $d \geq \sqrt{n} > d'$. Hence, the quantum code is degenerate.

If $p_i \equiv -1 \pmod{4}$, then by [15, Theorem 8], the permutation μ_{-1} gives a splitting for this code. Hence the odd-like distance must satisfy $d^2 - d + 1 \geq n$. ■

Example 7: Let us consider binary quantum duadic codes of length 7^m . Note that 2 is a quadratic residue modulo 7 as $4^2 \equiv 2 \pmod{7}$. Since $\text{ord}_7(2) = 3$ and $7 \parallel 2^3 - 1$, we have $z = 1$. By Theorem 6 for $m \geq 2$ there exist quantum codes with the parameters $[[7^m, 1, d]]_2$. As $p = 7 \equiv -1 \pmod{4}$ we have with $d^2 - d + 1 \geq 7^m$. But, d' , the distance of the (even-like) duadic codes is upper bounded by $p^z = 7$. Hence these codes are pure to $d' \leq 7$. Actually, using the fact that the true distance of the even-like codes is 4 [15] we can show that the quantum codes are pure to 4.

IV. QUANTUM DUADIC CODES – HERMITIAN CASE

Recall that if there exists an \mathbf{F}_{q^2} -linear $[n, k, d]_{q^2}$ code C such that $C^{\perp_h} \subseteq C$, then there exists an $[[n, 2k - n, \geq d]]_q$ quantum stabilizer code that is pure to d . In this section, we construct duadic quantum codes using this construction. Since $q^2 \equiv \square \pmod{n}$, duadic codes exist over \mathbf{F}_{q^2} for all n , when $\gcd(n, q^2) = 1$.

A. Basic Code Constructions

Lemma 8: Let C_i and D_i respectively be the even-like and odd-like duadic codes over \mathbf{F}_{q^2} , where $i \in \{0, 1\}$. Then $C_i^{\perp_h} = D_i$ if and only if there is a q^2 -splitting of n given by μ_{-q} , that is, $-qS_i \equiv S_{(i+1 \bmod 2)} \pmod{n}$.

Proof: See [12, Theorem 4.4]. ■

Lemma 9: Let $n = \prod p_i^{m_i}$ be an odd integer such that $\text{ord}_n(q)$ is odd. Then μ_{-q} gives a splitting of n over \mathbf{F}_{q^2} . In fact μ_{-1} and μ_{-q} give the same splitting.

Proof: Suppose that $\{S_0, S_1, a\}$ be a splitting. We know that each S_i is an union of some q^2 -ary cyclotomic cosets, so $q^2 S_i \equiv S_i \pmod{n}$. Now $q^{\text{ord}_n(q)} S_i \equiv S_i \pmod{n}$. If $\text{ord}_n(q) = 2k + 1$, then $q^{2k+1} S_i \equiv q S_i \equiv S_i \pmod{n}$; hence, μ_q fixes each S_i if the multiplicative order of q modulo n is odd.

Notice that if $\text{ord}_n(q)$ is odd, then $\text{ord}_n(q^2)$ is also odd. By [13, Lemma 5], we know that there exists a q^2 -splitting of n given by μ_{-1} if and only if $\text{ord}_n(q^2)$ is odd. Hence $-S_i \equiv S_{(i+1 \bmod 2)} \pmod{n}$. Since μ_q fixes S_i we have $-qS_i \equiv S_{(i+1 \bmod 2)} \pmod{n}$; hence, μ_{-q} gives a q^2 -splitting of n .

Conversely, if μ_{-q} gives a splitting of n , then $-qS_i \equiv S_{(i+1 \bmod 2)} \pmod{n}$. But as μ_q fixes S_i we have $-S_i \equiv$

$S_{(i+1 \bmod 2)} \pmod{n}$. Therefore μ_{-1} gives the same splitting as μ_{-q} . ■

Theorem 10: Let n be an odd integer such that $\text{ord}_n(q)$ is odd. Then there exists an $[[n, 1, d]]_q$ quantum code with $d^2 - d + 1 \geq n$.

Proof: By Lemma 9, there exist duadic codes $C_i \subseteq D_i$ with splitting given by μ_{-q} and μ_{-1} . This means that the $C_i \subseteq C_i^{\perp_h} = D_i$ by Lemma 8. Hence there exists an $[[n, n - (n - 1), d]]_q$ quantum code with $d = \text{wt}(D_i \setminus C_i)$. As μ_{-1} gives a splitting, we have $d^2 - d + 1 \geq n$ by Lemma 2. ■

B. Degenerate codes

We construct a family of degenerate quantum codes that has a large minimum distance.

Theorem 11: Let $n = \prod p_i^{m_i}$ be an odd integer with $\text{ord}_n(q)$ odd and every $p_i \equiv -1 \pmod{4}$. Let $t_i = \text{ord}_{p_i}(q^2)$, and $p_i^{z_i} \parallel q^{2t_i} - 1$. Then for $m_i > 2z_i$, there exist degenerate quantum codes with parameters $[[n, 1, d]]_q$ pure to $d' \leq \min\{p_i^{z_i}\} < d$ with $d^2 - d + 1 \geq n$.

Proof: From Lemma 5 we know that there exists an even-like duadic code with parameters $[n, (n - 1)/2, d']_{q^2}$ and $d' \leq \min\{p_i^{z_i}\}$.

Then by [15, Theorem 8], we know that for this code μ_{-1} gives a splitting. By Lemma 9, μ_{-q} also gives a splitting for this code.

Hence by Theorem 10 this duadic code gives a quantum duadic code $[[n, 1, d]]_q$, which is impure as $d' \leq \min\{p_i^{z_i}\} < \sqrt{n} < d$. ■

Finally, one can construct more quantum codes, for instance when $\text{ord}_n(q)$ is even, by finding the conditions under which μ_{-q} gives a splitting of n .

V. CONCLUSION

The motivation for this work was that many good quantum error-correcting codes, such as quantum MDS codes, are typically pure and thus require active corrective steps for all errors of small Hamming weight. At the other extreme are decoherence free subspaces (see [10], [16]) that do not require any active error correction at all, but perform poorly in terms of minimum distance. We pointed out that degenerate quantum codes can form a compromise, namely they can reach larger minimum distances while allowing at least some nice errors of low weight that do not require active error correction.

We have constructed two families of quantum duadic codes with the parameters $[[n, 1, \geq \sqrt{n}]]_q$ and have shown that they contain large subclasses of degenerate quantum codes. Though these codes encode only one qubit, they are interesting because they demonstrate that there exist families of classical codes which can give rise to remarkable degenerate quantum codes. Since these code are cyclic, we know that there exist several nice errors of small weight. A more detailed study of the weight distribution of classical duadic codes can reveal which code are particularly interesting for quantum error-correction. We note that generalizations of duadic codes, such as triadic and polyadic codes, can be used to obtain degenerate quantum codes with higher rates.

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